

Finite Codimensional Controllability for Evolution Equations

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Abstract

Motivated by infinite-dimensional optimal control problems with endpoint state constraints, in this Note, we introduce the notion of finite codimensional exact controllability for evolution equations. It is shown that this new controllability is equivalent to the finite codimensionality condition in the literatures to guarantee Pontryagin's maximum principle. As examples, LQ problems with fixed endpoint state constraints for a wave and a heat equation are analyzed, respectively. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Contrôle exacte co-dimensionnel fini pour des équations d'évolution. Motivé par des problèmes de contrôle optimal en dimension infinie avec des contraintes sur l'état final, nous introduisons la notion de contrôle exacte co-dimensionnel fini pour des équations d'évolution. On démontre que cette nouvelle notion de contrôlabilité est équivalente à la condition de codimensionalité finie qui garantit que le principe maximal de Pontryagin n'est pas trivial. A titre d'exemple, les problèmes LQ avec des contraintes d'état de point d'extrémité fixes sont analysés pour l'équation des ondes et l'équation de chaleur respectivement. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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1. Introduction

Controllability is one of the fundamental issues in control theory. Up to now, there are numerous works devoted to controllability problems of linear and nonlinear distributed parameter systems. In this Note,

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we will introduce a new concept on the finite codimensional exact controllability for linear control systems.

Let Y , Z and U be Hilbert spaces. Denote by $\mathcal{L}(Z; Y)$ the set of all bounded linear operators from Z to Y , by Y^* the dual space of Y , by $\text{span} D$ the closed subspace spanned by a subset D of Y , and by $\overline{\text{co}} D$ the convex closed hull of D . We identify U^* with U . Let $T > 0$ and $p \in (1, \infty]$. Write $\mathcal{U}_p = L^p(0, T; U)$. Consider the following linear control system:

$$y_t(t) = Ay(t) + F(t)y(t) + B(t)u(t), \quad t \in (0, T] \quad \text{and} \quad y(0) = y_0, \quad (1)$$

where u is the control variable and y is the state variable, $A: \mathcal{D}(A) \subset Y \rightarrow Y$ is a linear operator generating a C_0 -semigroup on Y , $F(\cdot) \in L^\infty(0, T; \mathcal{L}(Y; Y))$, $B(\cdot) \in L^\infty(0, T; \mathcal{L}(U; Y))$, and $y_0 \in Y$. For any $y_0 \in Y$ and $u(\cdot) \in \mathcal{U}_p$, (1) admits a unique mild solution $y(\cdot) \equiv y(\cdot; y_0, u(\cdot)) \in C([0, T]; Y)$. Define the reachable set of (1) as follows:

$$\mathcal{R}(T; y_0) = \{y(T; y_0, u(\cdot)) \in Y \mid y(\cdot) \text{ is the mild solution of (1) with some } u(\cdot) \in \mathcal{U}_p\}. \quad (2)$$

Let us first recall the notion of finite codimensionality.

Définition 1.1 *A subset M of Y is said to be finite codimensional in Y , if there exists an $x_0 \in \overline{\text{co}} M$ so that $\text{span}(M - \{x_0\})$ is a finite codimensional subspace of Y , and $\overline{\text{co}}(M - \{x_0\})$ has at least one interior point in this subspace.*

Now, we introduce the following new notion of finite codimensional exact controllability for (1).

Définition 1.2 *System (1) is said to be finite codimensional exactly controllable at the time T , if $\mathcal{R}(T; 0)$ is a finite codimensional subspace of Y .*

Recall that (1) is called exactly controllable at the time T , if $\mathcal{R}(T; 0) = Y$. Hence, the finite codimensional exact controllability defined above is clearly weaker than the exact controllability. In general, the finite codimensional exact controllability cannot be reduced to the usual exact controllability problem. Indeed, this can be done only for the very special case that $A + F(t)$ in (1) has an invariant subspace Y_0 , which is finite codimensional in Y and independent of $t \in [0, T]$.

The finite codimensional exact controllability is motivated by the study of optimal control problems with endpoint state constraints for infinite-dimensional systems. It is well known that, as a necessary condition for optimal controls, Pontryagin's maximum principle was established for very general finite-dimensional systems ([8]), which is one of the milestones in control theory. Nevertheless, very surprisingly, it fails for infinite-dimensional systems without further assumptions ([2], see also [5]). This leads to that for a long time, the Pontryagin maximum principle had been studied only for evolution equations without terminal state constraints. Until 1980s, by assuming a suitable finite codimensionality condition, [3,4,5] obtained the Pontryagin-type maximum principle for optimal control problems with endpoint constraints. However, it is usually quite difficult to verify this condition directly. In this Note, we reduce the finite codimensionality condition to a suitable finite codimensional exact controllability problem. By the duality technique, such a controllability problem is further reduced to some *a priori* estimate for its dual problem, which maybe is easily verified, at least for some nontrivial example.

We refer to [7] for a detailed proof of the results in this Note and other related results.

2. Main result

Let $\tilde{\mathcal{U}}$ be a bounded subset of the Banach space \mathcal{U}_p and $\overline{\text{co}} \tilde{\mathcal{U}}$ have at least one interior point. In the sequel, we choose

$$M = \{y(T) \in Y \mid y(\cdot) \text{ is the solution of (1) with } y_0 = 0 \text{ and some } u(\cdot) \in \tilde{\mathcal{U}}\}.$$

Also, we consider the following homogenous linear equation:

$$\phi_t(t) = -A^*\phi(t) - F(t)^*\phi(t), \quad t \in (0, T] \quad \text{and} \quad \phi(T) = \phi_T, \quad (3)$$

where $\phi_T \in X^*$, and A^* and $F(t)^*$ are respectively the dual operators of A and $F(t)$. Denote by C a generic positive constant, and by p' the Hölder conjugate of p . The main result of this Note is as follows.

Theorem 2.1 *The following assertions are equivalent:*

- (1) *The set M is finite codimensional in Y ;*
- (2) *The equation (1) is finite codimensional exactly controllable in Y ;*
- (3) *There is a finite codimensional subspace $\tilde{Y} \subseteq Y^*$ so that any solution ϕ of (3) satisfies*

$$|\phi_T|_{Y^*} \leq C|B(\cdot)^*\phi|_{L^{p'}(0,T;U)}, \quad \forall \phi_T \in \tilde{Y};$$

- (4) *There is a compact operator G from Y^* to a Banach space X so that any solution ϕ of (3) satisfies*

$$|\phi_T|_{Y^*} \leq C[|B(\cdot)^*\phi|_{L^{p'}(0,T;U)} + |G\phi_T|_X], \quad \forall \phi_T \in Y^*.$$

Theorem 2.1 can be applied to study optimal control problems with endpoint constraints for nonlinear distributed parameter systems. For concrete problems, as we shall see in the next section, one may use the fourth assertion in Theorem 2.1 to check the finite codimensional exact controllability of (1).

3. Two examples

This section is devoted to checking the finite codimensionality conditions in some LQ problems (with fixed endpoint constraints) for a wave and heat equations. Let Ω be a bounded domain in \mathbb{R}^N (for some $N \in \mathbb{N}$) with a smooth boundary Γ , and ω be a nonempty open subset of Ω . Denote by χ_ω the characteristic function of ω . Consider the following controlled wave and heat equations:

$$\begin{cases} y_{tt} - \Delta y + a(x, t)y = \chi_\omega u & \text{in } Q \equiv \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma \equiv \Gamma \times (0, T), \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } \Omega, \end{cases} \quad \text{and} \quad \begin{cases} y_t - \Delta y = \chi_\omega u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \quad (4)$$

where $u \in L^2(Q)$ is the control variable. In (4), $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $a(\cdot) \in L^\infty(Q)$ and $y^0 \in L^2(\Omega)$ are given. Also, for given targets $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $z^0 \in L^2(\Omega)$, set

$$\begin{aligned} \mathcal{U}_{ad}^1 &= \{u \in L^2(Q) \mid \text{The solution } y \text{ of the wave equation in (4) satisfies } (y(T), y_t(T)) = (z_0, z_1)\}, \\ \mathcal{U}_{ad}^2 &= \{u \in L^2(Q) \mid \text{The solution } y \text{ of the heat equation in (4) satisfies } y(T) = z^0\}, \end{aligned}$$

and

$$J(u(\cdot)) = \frac{1}{2} \int_Q [\alpha(x, t)y^2(x, t) + \chi_\omega \beta(x, t)u^2(x, t)] dx dt,$$

where $\alpha, \beta \in L^\infty(Q)$ are two given functions. Assume that \bar{u}_i is an optimal control, i.e., it satisfies that $J(\bar{u}_i(\cdot)) = \inf \{J(u(\cdot)) \mid u \in \mathcal{U}_{ad}^i\}$ ($i = 1, 2$). Write $\mathbb{B}_1 = \{u \in L^2(Q) \mid |u|_{L^2(Q)} \leq 1\}$ and

$$\begin{aligned} M_1 &= \{(y(T), y_t(T)) \mid y \text{ solves the wave equation in (4) with } (y_0, y_1) = (0, 0) \text{ and some } u \in \mathbb{B}_1\}, \\ M_2 &= \{y(T) \mid y \text{ solves the heat equation in (4) with } y^0 = 0 \text{ and some } u \in \mathbb{B}_1\}. \end{aligned}$$

Similar to the analysis in [6], if the sets M_1 and M_2 are finite codimensional accordingly in $H_0^1(\Omega) \times L^2(\Omega)$ and $L^2(\Omega)$, then one can obtain nontrivial necessary conditions for the optimal controls \bar{u}_i ($i = 1, 2$).

To verify this finite codimensionality condition, let us consider the following backward wave and heat equations:

$$\begin{cases} \psi_{tt} - \Delta\psi + a(x, t)\psi = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \Sigma, \\ (\psi(T), \psi_t(T)) = (\psi_1, \psi_2) \in L^2(\Omega) \times H^{-1}(\Omega), \end{cases} \quad \text{and} \quad \begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T \in L^2(\Omega). \end{cases} \quad (5)$$

By the fourth assertion in Theorem 2.1 and the known observability inequality for the wave equation in (5) with $a(\cdot) \equiv 0$ ([1]), we obtain the following positive result for the wave equation (with a rather general $a(\cdot)$) in (4).

Proposition 3.1 *For any $a(\cdot) \in L^\infty(Q)$, if (Ω, ω, T) fulfills the geometric optics condition (see [1]), then M_1 is finite codimensional in $H_0^1(\Omega) \times L^2(\Omega)$.*

By Theorem 2.1, Proposition 3.1 implies that under the geometric optics condition, the wave equation in (4) is finite codimensional exactly controllable. Notice that under the same condition, the exact controllability of the wave equation with a general coefficient $a(\cdot)$ is still an open problem.

Finally, by the fourth assertion in Theorem 2.1 again and the contradiction argument, we have the following negative result for the heat equation.

Proposition 3.2 *For any Ω , ω and $T > 0$, M_2 is not finite codimensional in $L^2(\Omega)$.*

By Proposition 3.2, the finite codimensionality condition fails for LQ problems for heat equations with fixed endpoint constraints.

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